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LHC SCHOTTKY SPECTRUM FROM MACRO-PARTICLE SIMULATIONS

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Abstract

We introduce a method for building Schottky spectra from macro-particle simulations performed with the PyHEADTAIL code, applied to LHC beam conditions. In this case, the use of a standard Fast Fourier Transform (FFT) algorithm to recover the spectral content of the beam becomes computationally intractable memory-wise, because of the relatively short bunch length compared to the large revolution period. This would imply having to handle an extremely large amount of data for performing the FFT. To circumvent this difficulty, a semi-analytical method was developed to compute efficiently the Fourier transform. The spectral content of the beam is calculated on the fly along with the macro-particle simulation and stored in a compact manner, independently from the number of particles, thus allowing the processing of one million macro-particles in the LHC, over 10 000 revolutions, in a few hours, on a regular computer. The simulated Schottky spectrum is then compared against theoretical formulas and measurements of Schottky signals previously obtained with lead ion beams in the LHC.

INTRODUCTION

Schottky signals contain information on various beam and machine parameters, such as momentum spread, betatron tune, synchrotron frequency and chromaticity, among others. The basic theory of Schottky signals predicts how all of these parameters express themselves in the recorded signal, after it is transformed to the frequency domain. This topic is treated in detail in a classical work by Boussard [1]. The developed theory is however limited only to the simplified beam dynamics and does not include, e.g., collective effects and beam interaction with the vacuum chamber through impedance. Due to the complex theoretical description of such effects, it is most suitable to study their impact on Schottky spectra using multi-particle simulations, as done in [2], where the effect of space charge was investigated.

Obtaining a Schottky spectrum from multi-particle simulations in the case of the CERN Large Hadron Collider (LHC), is particularly challenging computationally due to the highly sparse characteristic of the current signal produced by a single bunch. While discretising the current signal in time, and applying the FFT algorithm to retrieve the spectral content of the beam is manageable for smaller accelerators, typically having smaller revolution periods and longer bunches, like the CERN Proton Synchrotron (PS), this is unfortunately intractable memory-wise for the LHC. As a conservative estimate, for the LHC revolution frequency $f_0 \approx 11245.5$ Hz, if we sample a 1 ns long bunch with 100 points (which corresponds to a sampling frequency of 100 GHz) over 10 000 turns, we would have to store an array of $10^{11+4}/f_0 \sim 10^{11}$ signal samples[1] or around 1 TB of data, hence the need of an alternative approach.

The study presented herein is based on simulations performed with PyHEADTAIL [3, 4], a macro-particle code that can be used to track turn-by-turn the six-dimensional phase space evolution of a bunch, possibly including the effects of direct space charge and beam-coupling impedances (although this capability is not yet used for this study). The following section will present the method used to post-process the data from PyHEADTAIL and reconstruct the Schottky spectra, while the third section will compare the simulation results against theoretical formalisms and measurements from LHC ion beams.

METHOD

To simulate Schottky spectra one generates a set of macro-particles following an initial phase-space distribution, and then tracks turn-by-turn the macro-particles using the machine optics. The Schottky spectra are computed using an analytical Fourier transform, performed turn-by-turn at the same time as the tracking.

Longitudinal and Transverse Phase Space

An accurate description of the longitudinal phase space is necessary since the average Schottky spectrum explicitly depends on the distribution of synchrotron oscillation amplitudes [1]. For LHC lead ion bunches, the synchrotron oscillation amplitudes $\widehat{\tau}$ approximately follow a Rice distribution [5]. The amplitude of the momentum deviation $\widehat{\Delta p}$ can be deduced from $\widehat{\tau}$ by substituting Eq. (4) in Eq. (11) from [6], which yields

$$\widehat{\Delta p} = \frac{p_0}{|\eta|}\widehat{\tau}\left(1 - \frac{(h_{rf}\omega_0\,\widehat{\tau})^2}{16}\right)\Omega_0 \,,$$

with p_0 the reference momentum, η the slippage factor, Ω_0 the nominal angular synchrotron frequency[2], h_{rf} the radio frequency harmonic and ω_0 the angular revolution frequency of the LHC. The initial position with respect to the synchronous particle z and momentum deviation Δp of each

[1] Most of these will be zeros and while Sparse Fourier Transform algorithms exist, these are only applicable to sparse output signals.

[2] By nominal synchrotron frequency we mean the synchrotron frequency of the synchronous particle, for which the synchrotron amplitude vanishes.

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particle can then be computed from their synchrotron amplitude and phase[3] using Eqs. (10) and (12) of [6]:

$$z_i = \beta\, c\, \widehat{\tau_i}\, \cos(\phi_i), \qquad \Delta p_i = \widehat{\Delta p_i}\, \sin(\phi_i).$$

The initial transverse positions and momenta are instead obtained assuming a 2D Gaussian distribution in phase space.

Analytical Fourier Transform

Longitudinal Spectrum The current of a single particle i as seen from a pickup, can be expressed as

$$i_i(t) = q \sum_{n=0}^{N_t} \delta(t - nT_0 - \tau_{n,i}),$$

where δ is the Dirac delta function, N_t the number of turns, q the charge of the particle, $f_0 = 1/T_0$ the LHC revolution frequency, and $\tau_{n,i}$ the time difference between the particle i and the synchronous particle at turn n. The current produced by one bunch is simply obtained by summing the contributions of the N_p particles constituting the bunch:

$$i(t) = q \sum_{i=1}^{N_p} \sum_{n=0}^{N_t} \delta(t - nT_0 - \tau_{n,i}).$$

We proceed by computing analytically the Fourier transform

$$\widetilde{i(\omega)} = \int_{-\infty}^{\infty} i(t) e^{j\omega t} dt = q \sum_{n=0}^{N_t} e^{j\omega n T_0} \sum_{i=1}^{N_p} e^{j\omega \tau_{n,i}}. \quad (1)$$

To evaluate numerically this last expression we would have to store in memory the $N_t \times N_p$ numbers $\tau_{n,i}$ and to compute $O(N_t N_p N_f)$ exponentials, where N_f is the number of frequencies we want to plot. Both these requirements are highly limiting the number of particles we can simulate. We expect that the number of macro-particles required to make the PyHEADTAIL simulation realistic would be at least $N_p \sim 10^6$. Typical values for the number of turns and frequencies are $N_t \sim N_f \sim 10^4$ and come respectively from the acquisition time and spectral resolution of the LHC Schottky Monitor.

The spectrum in Eq. (1) can be computed more efficiently by replacing the right-hand exponential by its Taylor expansion. Since we are interested in the frequency domain of the LHC Schottky monitor working at the $h = 427725$ harmonic of the revolution frequency, the maximal value of the right-hand exponential argument will be $(\omega\tau)_{max} \approx 30$ (with $f_0 \approx 11.2455$ kHz and $\tau_{max} \sim 1$ ns) and will typically span over a maximum range of $|\Delta\omega|\ \tau_{max} \sim 10^{-5}$ for the sidebands satellites (e.g. for ~ 20 of them spaced by the nominal synchrotron frequency $f_s \sim 65$ Hz and for a fractional part of the tune $q_x \sim q_y \sim 0.3$). Expanding each exponential around $\omega_c \tau_{n,i}$ with $\omega_c = 2\pi h f_0$, instead of using the MacLaurin expansion, will allow the series to converge faster and to save some memory. The required number of terms, N_l, can be estimated by computing an upper bound for the rest of the Taylor expansion, which showed that $N_l = 4$ is typically sufficient for the simulation parameters stated above. Equation (1) is then replaced by

$$\begin{aligned}\widetilde{i(\omega)} &= q \sum_{n=0}^{N_t} e^{j\omega n T_0} \sum_{i=1}^{N_p} \sum_{l=0}^{N_l} \frac{e^{j\omega_c \tau_{n,i}}}{l!} (j\omega\tau_{n,i} - j\omega_c \tau_{n,i})^l \\ &= q \sum_{n=0}^{N_t} e^{j\omega n T_0} \sum_{l=0}^{N_l} \frac{j^l(\omega - \omega_c)^l}{l!} \sum_{i=1}^{N_p} e^{j\omega_c \tau_{n,i}} (\tau_{n,i})^l.\end{aligned}$$

By introducing the variables $\mathcal{L}_{n,l}$ and $\alpha_l(\omega)$ defined as

$$\mathcal{L}_{n,l} = \sum_{i=1}^{N_p} e^{j\omega_c \tau_{n,i}} (\tau_{n,i})^l, \qquad \alpha_l(\omega) = \frac{j^l(\omega - \omega_c)^l}{l!},$$

the previous expression can be written in the compact form

$$\widetilde{i(\omega)} = q \sum_{n=0}^{N_t} e^{j\omega n T_0} \sum_{l=0}^{N_l} \alpha_l(\omega)\ \mathcal{L}_{n,l}. \quad (2)$$

Computing all the $\mathcal{L}$ coefficients requires $O(N_t N_p)$ operations; the summation over N_l can be considered as $O(1)$, hence computing the spectrum thereafter takes $O(N_t N_f)$ operations. The total cost of evaluating Eq. (2) is then $O(N_t N_p)$ since $N_f << N_p$.

Transverse Spectrum A similar derivation can be performed for the transverse Schottky spectrum. With the dipole moment of a macro-particle i defined as $d_i(t) = x_i(t)\ i_i(t)$, we get a similar expression as Eq. (2):

$$\widetilde{d(\omega)} = q \sum_{n=0}^{N_t} e^{j\omega n T} \sum_{l=0}^{N_l} \alpha_l(\omega)\ \mathcal{T}_{n,l} \quad (3)$$

using the definition

$$\mathcal{T}_{n,l} = \sum_{i=0}^{N_p} x_{n,i}\ e^{j\omega_c \tau_{n,i}} (\tau_{n,i})^l,$$

where $x_{n,i} = x_i(t = nT_0 + \tau_{n,i}) \approx x_i(t = nT_0)$ is the horizontal transverse position of the particle i when passing in front of the detector at turn n.

Beside the considerable gain in computation time, another advantage of the formalism of Eqs. (2) and (3) is that the $\mathcal{L}$ and $\mathcal{T}$ coefficients can be calculated simultaneously with the tracking done in the PyHEADTAIL simulation. Hence there is no need to keep in memory the extremely large arrays of $\tau_{n,i}$, $x_{n,i}$ and $y_{n,i}$ for each turn and each particle - all the necessary information is contained in the $N_t \times N_l$ coefficients $\mathcal{L}$ and $\mathcal{T}$. Since the number of these coefficients does not depend on the number of macro-particles, we can increase the latter without any memory issue.

[3] Synchrotron phases are assumed to be uniformly distributed.

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Simulation Procedure

Figure 1 sketches the full simulation procedure. Note that we use the smooth approximation for the machine optics, and (unless stated otherwise) a non-linear RF system (hence leading to synchrotron frequency spread).

The spatial coordinates $x_{n,i}$, $y_{n,i}$ and $z_{n,i}$ of each marco-particle at each turn, obtained from PyHEADTAIL, are plugged into Eqs. (2) and (3) to reconstruct the longitudinal and transverse Schottky spectra. Finally, averaging these spectra over several random draws for the initial bunch, makes the characteristic Bessel satellites of the spectra appear. This final step is needed since the variance of the power spectral density of the non-central synchrotron satellites is proportional to the square of the number of simulated particles, due to the random phases of synchrotron motion. Therefore, adding more particles to the simulation procedure will not make the spectrum converge to its true average.

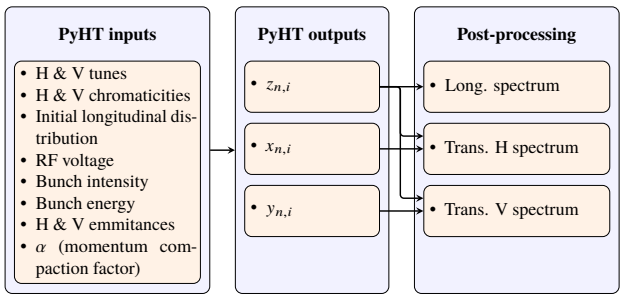

Figure 1: Schottky spectrum simulation procedure.

RESULTS

The method presented in the previous section is benchmarked against experimental Schottky spectra obtained with lead ion beams during LHC Run 2. The Schottky data consists of horizontal measurements of beam 2 taken at injection energy for the fill 7443.

Among the PyHEADTAIL input parameters described in Fig. 1, some are based on the fitting of the experimental spectrum as done in [5] and allows to determine precise machine parameters for the specific fill we want to reproduce. The parameters based on this fitting procedure are horizontal and vertical tunes, horizontal and vertical chromaticities, synchrotron amplitude distribution and nominal synchrotron frequency. The other parameters needed by PyHEADTAIL such as emmitances or bunch intensity and energy, come from direct measurements or machine design. Table 1 summarises the values of these parameters.

Table 1: PyHEADTAIL Simulation Parameters for Fill 7443

Intensity	1.76×10^8 ions per bunch
Energy per ion	36.9 TeV
Ion charge	82 e
Ion mass	193.687 GeV/c^2
ϵ_x, ϵ_y	1.5 µm
Tunes	$Q_x = 64.2827$, $Q_y = 59.2985$
Chromaticities	$Q'_x = 18.56$, $Q'_y = 11.64$
α	3.479×10^{-4}
h_{rf}	35640
RF voltage[4]	8.22 MV
LHC circumference	26.659 km
Rice parameters of $\hat{\tau}$ distribution	$\sigma = 1.306$ ns, $b = 0.216$

Figure 2 presents an overall view of the simulated longitudinal and transverse horizontal Schottky spectra obtained using Eqs. (2) and (3) respectively and plotted on the same frequency axis (frequencies in the plot have been shifted to the first harmonic). Figure 3 is a detailed view of these spectra compared against the experimental measurements of fill 7443. As visible from the plots, the agreement between simulations and measurements is very good. One can also observe that the effect of chromaticity, which makes the upper betatron sideband higher and thinner than the lower sideband, is well reproduced by the simulation.

[4] The RF voltage is computed from the fitted nominal synchrotron frequency $f_s = 64.67$ Hz with Eq. (110) of [7] and does not correspond to the measured RF voltage of 7.5 MV. This discrepancy is currently being investigated.

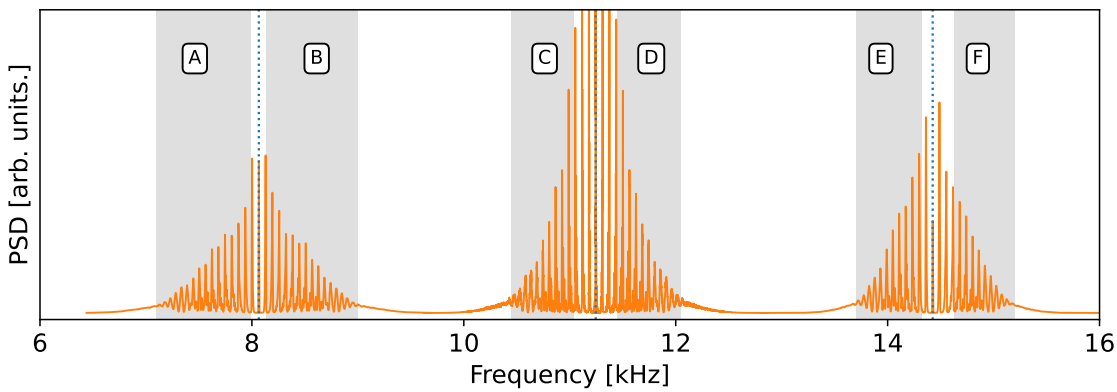

Figure 2: Simulated longitudinal (C, D) and transverse horizontal (A, B, E, F) Schottky spectra for the machine and beam configuration of LHC fill 7443. The dotted lines indicate respectively (from left to right): $(1 - Q_x) f_0$, f_0 and $(1 + Q_x) f_0$.

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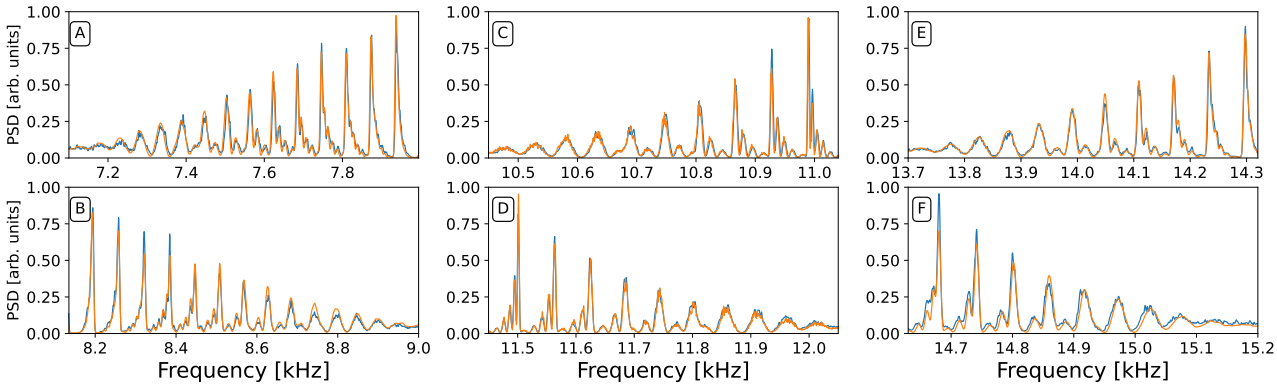

Figure 3: Measured (blue lines) longitudinal and transverse horizontal Schottky spectra compared with the simulated spectra (orange lines) for LHC fill 7443. The A–F region labels correspond to the frequency ranges of the shaded regions on Fig. 2.

Theoretical Comparison

Finally, we will compare the simulations against two theory-based methods suitable to reconstruct the Schottky spectrum. The first method is based on the Monte Carlo approach presented in [8]. It consists in drawing randomly, for each particle, a synchrotron oscillation time-amplitude $\widehat{\tau_i}$ and an initial synchrotron phase φ_{s_i} from a Rice and a uniform distribution respectively. Using the theoretical formula for the longitudinal spectrum of each particle i [1]

$$i_i(t) = q f_0 \sum_{n,p=-\infty}^{\infty} J_p(n\omega_0 \widehat{\tau_i})\, e^{j(n\omega_0 t + p\Omega_{s_i} t + p\varphi_{s_i})},$$

the Fourier transform of each individual particle can be directly deduced using the expresssion giving $\Omega_{s_i}(\widehat{\tau_i})$ from [6]. The Fourier transform being a linear operation, we can sum over the single-particle spectra in the complex plane using the sliding window method of [8] and taking the squared magnitude of this sum gives the Power Spectral Density (PSD) of the whole bunch (i.e. the Schottky spectrum). Finally, as done for PyHEADTAIL simulations, we need to average the PSD over several random draws for the initial bunch.

The transverse spectrum is obtained in a similar fashion by drawing an additional initial betatron phase φ_{β_i} from a uniform distribution and using the general expression for the dipole moment [9]

$$d_i(t) = q f_0 \frac{\widehat{x}}{2} \sum_{n,p=-\infty}^{\infty} J_p(\chi_{i,n}^{\pm}) e^{j(t[(n\pm Q)\omega_0 + p\Omega_{s_i}] \pm \varphi_{\beta_i} + p\varphi_{s_i})},$$

where

$$\chi_{i,n}^{\pm} = \left(n\widehat{\tau_i} \pm \frac{\widehat{Q_i}}{\Omega_{s_i}} \right) \omega_0.$$

Figure 4 shows a comparison between the results of our proposed simulation-based method, the Monte Carlo method, the matrix formalism developed in Refs. [5] and [9], and real measurements for two different regions of the Schottky spectrum. All the spectra have been normalised and are in very good agreement with each other.

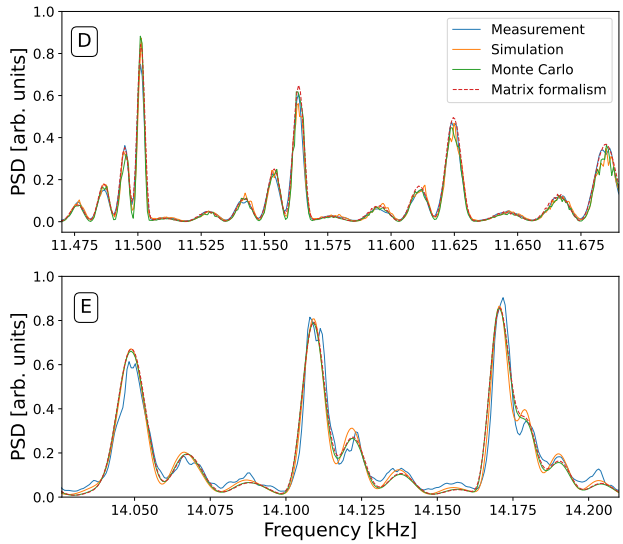

Figure 4: Comparison between measured, simulated and theory based longitudinal (D) and transverse horizontal (E) Schottky spectra for LHC fill 7443.

CONCLUSION

The aim of this work was to validate the development of a new method for calculating Schottky spectra from macro-particle simulations. This method will enable future studies on how effects such as beam-coupling impedances impact the measured spectra. After discussing the high computational resource requirements of traditional FFT algorithms when performing simulations for LHC beam and machine conditions, we showed how one can achieve a substantial reduction of its computational complexity. This method allows to closely reproduce the Schottky spectrum of a given LHC fill. The obtained results were shown to be in good agreement with reference measurements as well as with other theory-based methods, and reproduce the overall shape of the spectrum together with the detailed internal structure of the synchrotron satellites, as shown in Figs. 3 and 4.

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